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Holomorphic currents and duality in $N = 1$ supersymmetric theories

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Abstract: Twisted supersymmetric theories on a product of two Riemann surfaces possess non-local holomorphic currents in a BRST cohomology. The holomorphic currents act as vector fields on the chiral ring. The OPE's of these currents are invariant under the renormalization group flow up to BRST-exact terms. In the context of electric-magnetic duality, the algebra generated by the holomorphic currents in the electric theory is isomorphic to the one on the magnetic side. For the currents corresponding to global symmetries this isomorphism follows from 't Hooft anomaly matching conditions. The isomorphism between OPE's of the currents corresponding to non-linear transformations of fields of matter imposes non-trivial conditions on the duality map of chiral ring. We consider in detail the SU(N_c) SQCD with matter in fundamental and adjoint representations, and find agreement with the duality map proposed by Kutasov, Schwimmer and Seiberg.

Keywords: Duality in Gauge Field Theories, Supersymmetry and Duality.

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1. Introduction

Holomorphy is one of the most powerful tools in supersymmetric theories in four dimensions (for a review, see e.g. [1, 2]). It strongly constraints the phase structure and the nonperturbative behavior of the supersymmetric quantum systems. In particular, the current understanding of electric-magnetic duality is essentially based on the analysis of moduli spaces of vacua, chiral rings and holomorphic deformations of the superpotentials (see, e.g. [2]).

In this paper we consider aspects of supersymmetric theories that are not entirely controlled by chiral rings and holomorphy in the chiral field configuration space. More specifically, we consider twisted $N = 1$ supersymmetric theories with a conserved R-current on a Kähler manifold $M = \Sigma_1 \times \Sigma_2$, where Σ_1 (Σ_2) are two-dimensional Riemann surfaces. In general the twisted theory contains fields with fractional spins. We construct non-local holomorphic currents on Σ_1 (Σ_2) in a BRST cohomology [3]. These currents correspond to the conserved (generalized) currents of the untwisted theory. However, in contrast to the ordinary currents the coefficients in the operator product expansions (OPE) of the holomorphic currents are invariant under renormalization group flow and holomorphic in the coordinates on $\Sigma_1(\Sigma_2)$ up to BRST-exact terms. In particular, the central charge in the OPE of the holomorphic spin-2 current (associated with the stress tensor) with itself coincides with the *infrared* value of the a central charge, defined as the coefficient of the Euler density in the trace anomaly in the curved space. The central charge a is conjectured [4]–[13] to obey the four-dimensional analog of Zamolodchikov's c-theorem [14]. The holomorphic currents act as vector fields on the chiral ring. This fact has interesting consequences for electric-magnetic duality.

In the context of electric-magnetic duality the operator algebra generated by the OPE's of holomorphic currents in the electric theory has to be isomorphic to the one on the magnetic side. We show that for the currents corresponding to global symmetries of the theory this isomorphism follows from 't Hooft anomaly matching conditions. The isomorphism between OPE's of the currents corresponding to non-linear transformations of fields of matter imposes non-trivial conditions on the duality map of the chiral ring. We apply this construction to the SU(N_c) supersymmetric QCD [15] and the Kutasov-Schwimmer models [16, 17]. For these models there is a very simple duality map for holomorphic currents corresponding to global symmetries. In particular, the central charges of these currents coincide in the electric and magnetic descriptions of the theory. In the Kutasov-Schwimmer models we also construct holomorphic currents corresponding to *generalized* baryon currents that are associated with non-linear transformations of quark superfields. The action of these currents on the chiral ring agrees in a non-trivial way with the duality map for the chiral operators proposed in [16, 17, 18]. This observation provides an additional check of the duality in the Kutasov-Schwimmer models.

The paper is organized as follows. In section 2 we review and generalize the twisting of $N = 1$ supersymmetric theories with a conserved R-current on Kähler manifold. In section 3 we define non-local currents that are holomorphic in the BRST cohomology and discuss their OPE algebra. In section 4 we apply this construction to the SU(N_c) SQCD with matter in fundamental and adjoint representations. We conclude by summarizing the results of the paper. Some possible generalizations of these results are also discussed.

2. Twisted $N = 1$ supersymmetric theory

In [19] it has been shown that a supersymmetric theory on a Kähler manifold M can be twisted to preserve one of the supersymmetry generators. Such a twisting is essentially equivalent to coupling of the theory to an appropriate external vector U(1) field V_μ, that enters the same $N = 1$ supergravity multiplet with the veirbein field e^a_μ. This vector field effectively changes spins of the quantum fields. The unbroken supersymmetry generator becomes a BRST generator Q of the twisted theory. For consistency at the quantum level one has to ensure that the vector field V_μ is coupled to a non-anomalous conserved R-current.

In this paper we focus on the theories twisted on $M = \Sigma_1 \times \Sigma_2$, where $\Sigma_{1,2}$ are Riemann surfaces. The metric on M is chosen to be block diagonal $g = g_{\Sigma_1} \oplus g_{\Sigma_2}$. We denote the complex coordinates on Σ_1 and Σ_2 as z ($\bar{z}$) and w ($\bar{w}$) respectively. The holonomy group of M is $\mathrm{U}(1) \times \mathrm{U}(1)$. This allows us to consider a partial twisting [20]. For example, we can choose $\Sigma_1 = T^2$ a torus with a flat metric and keep the fields untwisted along Σ_1 and twisted along Σ_2. However, to simplify notations we will consider the fully twisted version of the theory.

Consider a supersymmetric gauge theory with a gauge group G and chiral superfields Φ_i in r_i representations of the gauge group. The gauge multiplet of the twisted theory contains a gauge field $A_\mu = \{A_z, A_w, A_{\bar{z}}, A_{\bar{w}}\}$, a scalar fermion $\bar{\lambda}$ and a fermionic $(\bar{1}, \bar{1})$-differential $\bar{\lambda}_{\bar{z}\bar{w}}$, a fermionic $(1, 0)$-differential χ_z and a fermionic $(0, 1)$-differential χ_w, and an auxiliary scalar field D.

The R-charges of the chiral superfields in the untwisted theory may be fractional. Therefore, the twisted theory may contain fields with fractional spins. In order to determine the spin content of a twisted supermultiplet Φ it is instructive to notice that the R-current is a linear combination of the canonical R-current (corresponding to phase rotations of the Grassmann coordinates θ of the untwisted theory) and flavor currents. A chiral multiplet twisted with the canonical R-current [3] contains a scalar bosonic field ϕ ($\bar\phi$), a scalar fermion $\bar\psi$, the fermionic fields $\psi_{\bar z}$, $\psi_{\bar w}$ and $\bar\psi_{zw}$ which are $(\bar 1,0)$, $(0,\bar 1)$ and $(1,1)$-differentials respectively, and the auxiliary bosonic fields transforming $N_{\bar z\bar w}$ and $\bar N_{zw}$ as $(\bar 1,\bar 1)$ and $(1,1)$-differentials respectively.

An additional twist with $2j$ positive flavor charge corresponds to an additional twist of ϕ and $\psi_{\bar z(\bar w)}$ with K^j and $\bar\phi$ and $\bar\psi$, $\bar\psi_{zw}$ with K^{-j}, where K stands for the canonical bundle over Σ_2. This follows from the fact that by twisting a field with one unit of R-charge its spin changes by $1/2$. We will assume that K^j and K^{-j} are well defined line bundles over Σ_2 and hence, $j\chi$ is an integer, where χ stands for the Euler characteristic of Σ_2. As a result, under the twisting the fields become differentials of fractional degrees as follows

$$\begin{aligned}
&\phi \to (j,j)\,, \quad \psi_{\bar w} \to (j,\overline{1-j})\,, \quad \psi_{\bar z} \to (\overline{1-j},j)\,, \quad N_{\bar z\bar w} \to (\overline{1-j},\overline{1-j})\,,\\
&\bar\phi \to (\bar j,\bar j)\,, \quad \bar\psi \to (\bar j,\bar j)\,, \quad \bar\psi_{zw} \to (1-j,1-j)\,, \quad \bar N_{zw} \to (1-j,1-j)\,.
\end{aligned} \tag{2.1}$$

Here (j_1,j_2) stands for a differential of the degrees j_1 and j_2 on the Σ_1 and Σ_2 respectively. In these notations a differential ω on a surface Σ of a degree j ($\bar j$) transforms as $\omega(z,\bar z)dz^j \to \omega(z',\bar z)dz'^j$ ($\omega(z,\bar z)d\bar z^j \to \omega(z',\bar z)d\bar z'^j$) under $z\to z'$, $\bar z\to\bar z'$.

In order to simplify notations, we do not show explicitly the degrees of the differentials. Instead, we will keep notations corresponding to the canonical twisting so that the additional twist is implemented by a non-trivial external U(1) field.

The twisted lagrangian in the theory without any superpotential reads as follows [19]:

$$\begin{aligned}
L = &\frac{1}{e^2}\sqrt{g}\left\{Q,\mathrm{Tr}\left[-i\bar\lambda g^{z\bar z}F_{z\bar z} - i\bar\lambda g^{w\bar w}F_{w\bar w} + \frac{1}{2}D\bar\lambda - \frac{i}{2}\bar\lambda_{\bar z\bar w}F^{\bar z\bar w}\right]\right\} +\\
&+\sum_i \sqrt{g}\left\{Q, -\frac{1}{2}\bar\psi^{\bar z\bar w,i}N^i_{\bar z\bar w} + \bar\phi^i(D^{\bar z}\psi^i_{\bar z} + D^{\bar w}\psi^i_{\bar w}) - i\bar\phi^i\bar\lambda\phi^i\right\},
\end{aligned} \tag{2.2}$$

where we left color indices implicit and F stands for the strength tensor of the gauge field, e^2 is the gauge coupling. Here Q is the scalar nilpotent generator of BRST transformations which for the gauge multiplet read

$$\begin{aligned}
&QA_z = \chi_z\,, \quad QA_w = \chi_w\,, \quad QA_{\bar z} = QA_{\bar w} = 0\,, \quad Q\chi_z = Q\chi_w = 0\,,\\
&Q\bar\lambda = -D\,, \quad Q\bar\lambda_{\bar z\bar w} = 2iF_{\bar z\bar w}\,, \quad QD = 0
\end{aligned} \tag{2.3}$$

while for the chiral multiplet Φ^i we have

$$\begin{aligned}
&Q\phi^i = 0\,, \quad Q\psi^i_{\bar w} = D_{\bar w}\phi^i\,, \quad Q\psi^i_{\bar z} = D_{\bar z}\phi^i\,, \quad QN^i_{zw} = D_{\bar z}\psi^i_{\bar w} - D_{\bar w}\psi^i_{\bar z} + \frac{1}{2}\bar\lambda_{\bar z\bar w}\phi^i\,,\\
&Q\bar\phi^i = \bar\psi^i\,, \quad Q\bar\psi^i = 0\,, \quad Q\bar\psi^i_{zw} = -2\bar N^i_{zw}\,, \quad Q\bar N^i_{zw} = 0\,,
\end{aligned} \tag{2.4}$$

where $D_{z(w)}$ and $D_{\bar z(\bar w)}$ stand for covariant derivatives.

The ghost numbers can be taken equal to the R-charges of the fields. Note the expression (2.2) is Q-exact. Therefore, one can perform computations of OPE's of Q-closed operators in the weak coupling limit $e^2 \to 0$.

In the case of a non-vanishing superpotential $W \neq 0$ the contribution to the lagrangian of the superpotential terms reads as

$$L_{sp} = \sqrt{g} \left[\sum_{i,j} \psi_{\bar{z}}^i \psi_{\bar{w}}^j \partial_i \partial_j W(\phi) + \frac{1}{2} \sum_i N_{\bar{z}\bar{w}}^i \partial_i W(\phi) + \left\{ Q, \sum_i \bar{\psi}_{zw}^i \partial_i W(\bar{\phi}) \right\} \right] . \qquad (2.5)$$

Here the gauge indices are implicit, and we assume that the superpotential W does not violate the R-symmetry. Therefore, the expression in the brackets transforms as a scalar. One can check that $\{Q, L_{sp}\} = 0$. Note that the last term in eq. (2.5) is Q-exact and can be neglected in computations in the Q-cohomology. The chiral ring of the untwisted theory maps to a ring of local observables that are invariant under Q. If $H^{2,0}(M) \neq 0$ then it is also possible to construct non-local Q-closed operators as integrals over holomorphic 2-cycles in M [3]. However, these observables are irrelevant for the discussion below.

3. Holomorphic currents

In [3] it was shown that in the twisted theory one can define BRST-closed currents on Σ_1 that are holomorphic up to Q-exact terms. These currents generate chiral algebras[1] on Σ_1. Here we consider a generalization of this construction. Let us define a spin 1 current on Σ_1 (we omit the dependence on external metric)

$$j_z(g) = 2\pi \sum_i \int_{\Sigma_2} \left(-D_z \bar{\phi}^i g^i + \sum_j \bar{\psi}_{zw}^i (\partial_j g^i) \psi_{\bar{w}}^j \right) . \qquad (3.1)$$

Here $g^i = g^i(\phi)$ are arbitrary functions of ϕ^k transforming as $R_i/2$-differentials on Σ_2 (also assume that g^i do not depend on coordinates explicitly). We also assume that g^i transform according to the representation r_i of the gauge group. By imposing the condition of the Q-closeness of $j_z(g)$ and using the equations of motion one can easily get

$$\int_{\Sigma_2} \sum_i \sum_j \partial_j (g^i \partial_i W) \psi_{\bar{w}}^j = 0 . \qquad (3.2)$$

Therefore, this condition is satisfied for

$$\sum_i g^i \partial_i W = 0 . \qquad (3.3)$$

This condition means that the superpotential W is invariant under transformations $\phi^i \to \phi^i + g^i(\phi)$.

[1] Chiral "projected algebras" have been also discussed in [21] in more general context of higher-dimensional analogues of the bc system of $2D$ RCFT.

In general, the condition $Qj_z(g) = 0$ may be also violated by anomalies. The condition of absence of anomalies can be obtained at the one-loop level [3] and reads

$$\sum_i (\partial_i g^i) T(r_i) = 0\,, \tag{3.4}$$

where $T(r_i)$ stands for the Dynkin index of the r_i representation of the gauge group. An important property of the current $j_z(g)$ is that it is holomorphic in the Q-cohomology, i.e.

$$\partial_{\bar z} j_z(g) = \{Q, \ldots\}\,, \tag{3.5}$$

where $\{Q, \ldots\}$ stands for Q-exact terms. This fact can be checked by using the equations of motion. eq. (3.5) may also suffer from an anomaly. However, this anomaly exactly cancels for (3.4).

The coefficients in the OPE of two currents $j_z(g)$ and $j_z(g')$ are invariant under the renormalization group modulo Q-exact terms [3]. Notice that the lagrangian of the theory is a sum of various Q-exact terms and the Q-closed chiral contribution from the superpotential. The superpotential is not renormalizable (see, e.g. [2]). Therefore, the dependence on a renormalization group scale may appear only from Q-exact terms in the lagrangian that may contribute only Q-exact terms to the OPE.

Let us consider the OPE of $j_z(g)(z,\bar z)$ with an element of the chiral ring given by a polynomial $P(\phi)$. We have

$$\begin{aligned} j_z(g)(z,\bar z)P(u,\bar u,v,\bar v) &= -2\pi\int_{\Sigma_2} d^2w \sum_i g^i(z,\bar z,w,\bar w)\partial_k P(u,\bar u,v,\bar v)\times \\ &\quad \times\langle D_z\bar\phi^i(z,\bar z,w,\bar w)\phi^k(u,\bar u,v,\bar v)\rangle + \cdots \\ &= -2\pi\int_{\Sigma_2} d^2w \sum_i g^i\partial_i P\,\frac{1}{4\pi^2}\partial_z\frac{1}{|z-u|^2+|w-v|^2} + \cdots \\ &= -\frac{1}{2}\sum_i g^i\ \partial_i P\frac{1}{z-u} + \cdots\,. \end{aligned} \tag{3.6}$$

Here $z, \bar z$ and $u, \bar u$ stand for coordinates on Σ_1, and $w, \bar w$ and $v, \bar v$ are coordinates on Σ_2. In eq. (3.6) we omitted Q-exact and regular terms. One can see that the current $j_z(g)$ acts as a vector field in the chiral ring generated by polynomials of chiral fields Φ^i.

As shown in [3] the holomorphic currents j^a corresponding to a flavor group G_F generate the Kac-Moody algebra (in this case the functions g^i are linear in the fields of matter)

$$j^a_z(z,\bar z)j^b_z(0,0) = \frac{k\delta^{ab}}{z^2} + \frac{if^{abc}}{z}j^c(0,0) + \text{regular and } Q - \text{exact terms} \tag{3.7}$$

where the central charge $k \propto (1-h)$ is constant and h is the genus of Σ_2.

In general, for non-linear functions g^i the algebra generated by the OPE is more complicated. Here we focus only on the most singular "central" terms. In the limit of vanishing gauge coupling one has

$$\begin{aligned} j_z(g)(z,\bar z)j_z(g')(u,\bar u) &= (2\pi)^2\int_{\Sigma_2} d^2w\int_{\Sigma_2} d^2v\sum_{i,j}(\partial_j g^i)(z,w)(\partial_i g'^j)(u,v)\times \\ &\quad \times(\langle\phi^j(z,w)\partial_u\bar\phi^i(u,v)\rangle\langle\phi^i(u,v)\partial_z\bar\phi^j(z,w)\rangle - \\ &\quad -\langle\psi^j_{\bar w}(z,w)\bar\psi^i_{uv}(u,v)\rangle\langle\psi^i_{\bar v}(u,v)\bar\psi^j_{zw}(z,w)\rangle) + \cdots\,. \end{aligned} \tag{3.8}$$

The propagators read as follows

$$
\begin{aligned}
\langle \phi^j(z,w)\bar{\phi}^i(u,v)\rangle &= \frac{\delta^{ij}}{\Delta_{\Sigma_1}^{(\frac{R_i}{2})}+\Delta_{\Sigma_2}^{(\frac{R_i}{2})}}\delta^2(z-u)\delta^2(w-v)\,,\\
\langle \psi^j_{\bar w}(z,w)\psi^i_{\bar v}(u,v)\rangle &= \frac{\delta^{ij}}{\Delta_{\Sigma_1}^{(\frac{R_i}{2})}+\Delta_{\Sigma_2}^{(1-\frac{R_i}{2})}}\delta^2(z-u)\delta^2(w-v)\,,
\end{aligned}
$$

where $\Delta_{\Sigma}^{(r)}$ stands for the Laplace operator on Σ for differentials of degree r. The dependence of the chiral fields ϕ^i on antiholomorphic coordinates $\bar w$ and $\bar v$ can be dropped since $D_{\bar w}\phi^i$ is Q-exact. However, in the presence of external metric the dependence on holomorphic coordinates w and v is important.

By expanding g^i in powers of $w-v$ one gets contributions of local operators of different dimensions. For the contribution of the operator of the lowest dimension one can drop the dependence of g and g' on the holomorphic coordinates. In this case the contributions of the non-zero modes of the Laplace operators on Σ_2 to eq. (3.8) are canceled due to the underlying supersymmetry. The contributions of zero modes give

$$
-\frac{1}{(z-u)^2}\sum_{i,j}\mathrm{index}\left(\frac{R_i}{2}\right)\partial_j g^i\partial_i g'^j = -\frac{\chi}{2(z-u)^2}\sum_{i,j}(\partial_j g^i)(\partial_i g'^j)(1-R_i)\,, \qquad (3.9)
$$

where index$(R_i/2)$ stands for the index of the Dirac operator acting on a r-differential on Σ_2, and $\chi = 2(1-h)$ is the Euler characteristic of Σ_2. In eq. (3.9) we used the Riemann-Roch formula [22] to compute index(r)

$$
\mathrm{index}(r) = h^{0,r} - h^{0,1-r} = (1-2r)(1-h)\,,
$$

where h is a genus of Σ_2. The "central" operator in eq. (3.9) is an element of the chiral ring. In particular, for linear functions $g^i = q_i\phi^i$ and $g'^i = {q'}_i\phi^i$ the currents $j_z(g)$ and $j_z(g')$ correspond to $\mathrm{U}(1)_q$ and $\mathrm{U}(1)_{q'}$ flavor groups, and $q(\phi^i) = q_i$ and $q'(\phi^i) = q'_i$. In this case one gets the central term

$$
-\frac{\chi}{2(z-u)^2}\sum_i q_i q'_i(1-R_i)
$$

that is proportional to the 't Hooft anomaly $\langle qq'R\rangle$.

The current $j_z(g)$ originates from a conserved 4-dimensional current corresponding to a chiral field transformation[2]

$$
\delta\phi^i = g^i(\phi)\,. \qquad (3.10)
$$

For non-linear functions g^i the transformation (3.10) does not generate any global symmetries. Therefore, in contrast to the currents corresponding to the global symmetries in general there is no obvious prescription for the duality map of generalized currents in the untwisted theory.

[2]The important role of generalized currents corresponding to non-linear chiral field transformations was pointed out in [23] where generalized Konishi anomalies are used to determine the quantum chiral ring.

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An ordinary current is quadratic in fields of matter

$$j_\mu = \sum_i q_i(\bar\phi_i D_\mu\phi^i - D_\mu\bar\phi_i\phi^i + \bar\psi_i\sigma_\mu\psi^i)\,,$$

where q_i are the charges of the chiral fields ϕ^i and corresponds to $j_z(g)$ with linear functions g^i. In general, the central charges of the conserved flavor currents are not invariant under the renormalization group flow[3]

$$\langle j_\mu(x)j_\nu(0)\rangle = \frac{1}{16\pi^4}(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)\left(\frac{b(e(1/x))}{x^4}\right),$$

where e stands for coupling constants. This fact does not contradict the invariance of the OPE of $j_z(g)$. The difference between $j_z(g)$ and the z-component of j_μ is proportional to $\partial_z(\bar\phi_i g^i)$, and $\bar\phi_i g^i$ is the lowest component of the supermultiplet containing the current j_μ. This difference is responsible for the different renormalization properties.

Following [3] we also define a spin 2 current related to the stress tensor

$$\begin{aligned} T_{zz} &= T^{\text{gauge}}_{zz} + T^{\text{matter}}_{zz} + \partial_z j_z(g)\,, && (3.11)\\ T^{\text{gauge}}_{zz} &= \frac{2\pi}{e^2}\int_{\Sigma_2}\text{Tr}[g^{w\bar w}F_{zw}F_{z\bar w} - iD_z\bar\lambda\chi_z]\,, \\ T^{\text{matter}}_{zz} &= 2\pi\int_{\Sigma_2} -D_z\bar\phi_i D_z\phi^i + g^{z\bar z}\bar\psi_{zw,i}D_z\psi^i_{\bar w}\,. && (3.12)\end{aligned}$$

By imposing the condition of Q-closeness $\{Q, T_{zz}\} = 0$ and using equations of motion one gets

$$W + \sum_j g^j\partial_j W = 0\,. \tag{3.13}$$

The condition of cancellation of anomaly gives

$$-C_2(G) + \sum_i T(r_i)(1 + 2\partial_i g^i) = 0\,. \tag{3.14}$$

Various solutions to these constraints differ by a z-derivative of a holomorphic current corresponding to a conserved current of the untwisted theory. A particular solution is

$$g^i = -\frac{1}{2}R_i\,, \tag{3.15}$$

where R_i stands for the R-charge of the superfield Φ_i. The operator T_{zz} generates a Virasoro algebra in the Q-cohomology [3]:

$$T(z,\bar z)T(w,\bar w) = \frac{c}{2(w-z)^4} + \frac{2T}{(z-w)^2} + \frac{\partial_z T}{z-w} + \text{regular terms} + \{Q,\ldots\}\,. \tag{3.16}$$

As discussed above the central charge c of this Virasoro algebra can be computed by evaluating appropriate one-loop diagrams. Moreover, the contributions of the non-zero

[3] However, in many cases the values of the central functions b can be computed at the infrared limit, e.g. at the conformal fixed points [24].

modes are canceled due to the underlying supersymmetry. The contributions of zero modes on Σ_2 in the denominators of the propagators give:

$$c = -\chi \mathrm{dim} G + 2 \sum_i \mathrm{dim}\ r_i \left(1 - 3R_i + \frac{3}{2}R_i^2\right) \mathrm{index}\left(\frac{R_i}{2}\right). \tag{3.17}$$

The first term in eq. (3.17) corresponds to the contribution of the gauge sector [3] while the second one comes from diagrams with propagators of field of matter. It is expedient to rewrite the expression for the central charge as follows

$$\begin{aligned} c &= \chi \left[-\mathrm{dim} G + \sum_i \mathrm{dim} r_i \left(1 - 3R_i + \frac{3}{2}R_i^2\right)(1 - R_i)\right] \\ &= \frac{\chi}{2}\left[-2\mathrm{dim} G + \sum_i (3(1-R_i)^3 - (1-R_i))\right] \\ &= \frac{\chi}{2}(-3\langle R^3\rangle + \langle R\rangle)\,. \end{aligned} \tag{3.18}$$

Here $\langle R^3\rangle$ and $\langle R\rangle$ stand for the corresponding coefficients for 't Hooft anomalies for $\mathrm{U}(1)_R^3$ and $\mathrm{U}(1)_R$ respectively. For a different choice of $j_z(g)$ in eq. (3.11) the value of the central charge generically would involve other 't Hooft anomalies. Notice that eq. (3.18) is proportional to the infrared value of the a central charge of the stress tensor provided the twisting R-current belongs to the $N = 1$ superconformal multiplet at the fixed point of the renormalization group [10] (for a recent discussion see [25]).

4. Electric-magnetic duality

In this section we apply the above construction to theories related by electric-magnetic duality. Since the coefficients in the OPE's of the holomorphic currents are invariant under renormalization group flows the algebras generated by the OPE's have to be isomorphic in electric and magnetic theories. In particular, the central charges have to coincide in the theories related by duality. This isomorphism provides an additional test for electric-magnetic dualities. However, in order to establish such an isomorphism one has to specify a map between holomorphic currents in the theories related by duality.

4.1 $\mathrm{SU}(N_c)$ theory with fundamental flavors

Consider the $\mathrm{SU}(N_c)$ supersymmetric gauge theory with N_f flavors of the chiral superfields Q_i and $\tilde{Q}^i$, $i = 1, \ldots, N_f$ in the fundamental and antifundamental representation respectively, $N_f \geq N_c$. The anomaly free global symmetry is

$$G_f = \mathrm{SU}(N_f) \times \mathrm{SU}(N_f) \times \mathrm{U}(1)_B \times \mathrm{U}(1)_R\,. \tag{4.1}$$

The quark fields transform under (4.1) as

$$Q : \left(N_f, 1, 1, 1 - \frac{N_c}{N_f}\right), \qquad \tilde{Q} : \left(\bar{N}_f, -1, 1 - \frac{N_c}{N_f}\right).$$

The chiral ring is generated by the following gauge invariant operators

$$\begin{aligned} M^i_j &= Q^i \tilde{Q}_j \,, \\ B^{[i_1 \dots i_{N_c}]} &= Q^{i_1} \cdots Q^{i_{N_c}} \,, \\ \tilde{B}_{[j_1 \dots j_{N_c}]} &= \tilde{Q}_{j_1} \cdots \tilde{Q}_{j_{N_c}} \,. \end{aligned}$$

In the infrared limit this theory has a dual (magnetic) description [15] in terms of the $\mathrm{SU}(N_f - N_c)$ supersymmetric theory with N_f fundamental flavors q_i and $\tilde{q}^i$, $i = 1, \dots, N_f$, and a color-singlet meson field M^j_i, $i, j = 1, \dots, N_f$. The magnetic theory has a superpotential

$$W_{\mathrm{mag}} = q_i M^i_j \tilde{q}^j \,.$$

The fields in the magnetic theory transform as

$$\begin{aligned} q &: \left(\bar{N}_f, 1, \frac{N_c}{N_f - N_c}, \frac{N_c}{N_f} \right), \\ \tilde{q} &: \left(1, N_f, \frac{N_c}{N_f - N_c}, \frac{N_c}{N_f} \right), \\ M &: \left(N_f, \bar{N}_f, 0, 2(1 - \frac{N_c}{N_f}) \right). \end{aligned}$$

The chiral ring is generated by M^i_j and baryons defined similarly with the electric theory.

In the twisted electric theory one can define a holomorphic current corresponding to the baryon symmetry $\mathrm{U}(1)_B$

$$j^B_z[Q, \tilde{Q}] = 2\pi \int_{\Sigma_2} d^2 w \sum_{i=1}^{N_f} \left[\left(-\partial_z \bar{Q}_i Q^i + \bar{\psi}_{i,zw} \psi^i_{\bar{w}} \right) - \left(-\partial_z \bar{\tilde{Q}}^i \tilde{Q}_i + \bar{\tilde{\psi}}^i_{zw} \tilde{\psi}_{i,\bar{w}} \right) \right]$$

where we use the same notations for the bosonic components of the chiral supermultiplets Q^i and $\tilde{Q}_i$ as for the corresponding supermultiplets, and $\psi^i_{\bar{w}}$, $\bar{\psi}_{i,zw}$ ($\tilde{\psi}_{i\bar{w}}$, $\bar{\tilde{\psi}}^i_{zw}$) stand for fermionic superpartners of $Q^i(\tilde{Q}_i)$ respectively. In a similar way, one can define the baryon current in the magnetic theory

$$\frac{N_c}{N_f - N_c} j^B_z[q, \tilde{q}] \,.$$

The coefficient $N_c/(N_f - N_c)$ is introduced for the correct normalization of the current.

It is easy to check that these currents are Q-closed and holomorphic in the Q-cohomology (see eqs. (3.3) and (3.5)). Moreover, they act on the baryon operators in the same way in the both theories and correctly compute the baryon charges

$$j^B_z(z, \bar{z}) \; B^{[i_1 \dots i_{N_c}]}(u, \bar{u}) = -\frac{1}{2(z-u)^2} N_c B^{[i_1 \dots i_{N_c}]}(u, \bar{u}) + \cdots \,.$$

Therefore, we conclude

$$j^B_z[Q, \tilde{Q}] = j^B_z[q, \tilde{q}] + Q\text{-exact terms} \,.$$

As discussed in the previous section the central charge of this current is proportional to the corresponding 't Hooft anomaly $\langle \mathrm{U}(1)_B^2\,\mathrm{U}(1)_R\rangle = -2N_c^2$ that has the same value in both theories. In a similar way, one can identify the $\mathrm{SU}(N_f)$ and the spin-2 T_{zz} currents in the electric and magnetic theories and check the matching of the corresponding central charges.

4.2 $\mathrm{SU}(N_c)$ theory with fundamental flavors and a chiral superfield in adjoint representation

The Kutasov-Schwimmer model [16, 17] is more interesting for study of holomorphic currents. In this case, one has a $\mathrm{SU}(N_c)$ gauge theory with N_f fundamental flavors, Q^i, $\tilde{Q}_i$, $i = 1, \ldots, N_f$ and a chiral superfield X in the adjoint representation of the gauge group with a superpotential

$$W_{el} = \frac{s_0}{k+1}\,\mathrm{Tr}\,X^{k+1}\,, \qquad k \geq 1\,, \tag{4.2}$$

where s_0 is a complex parameter, $kN_f \geq N_c$. The anomaly free global symmetry group given by (4.1). The matter fields transform under (4.1) as:

$$\begin{aligned}
Q:&\quad \left(N_f, 1, 1, 1 - \frac{2}{k+1}\frac{N_c}{N_f}\right),\\
\tilde{Q}:&\quad \left(1, \bar{N}_f, 1, 1 - \frac{2}{k+1}\frac{N_c}{N_f}\right),\\
X:&\quad \left(1, 1, 0, \frac{2}{k+1}\right).
\end{aligned}$$

The dual (magnetic) theory has gauge group $\mathrm{SU}(kN_f - N_c)$, N_f fundamental flavors, q_i, $\tilde{q}^i$, an adjoint field Y, and gauge singlets M_j, $j = 1, \ldots, k$ with the superpotential

$$W_{\mathrm{mag}} = -\frac{s_0}{k+1}\,\mathrm{Tr}\,Y^{k+1} + \frac{s_0}{\mu^2}\sum_{j=1}^{k} M_j \tilde{q} Y^{k-j} q\,. \tag{4.3}$$

The transformation properties of the magnetic matter fields under (4.1) are:

$$\begin{aligned}
q:&\quad \left(\bar{N}_f, 1, \frac{N_c}{\bar{N}_c}, 1 - \frac{2}{k+1}\frac{\bar{N}_c}{N_f}\right),\\
q:&\quad \left(1, N_f, \frac{N_c}{\bar{N}_c}, 1 - \frac{2}{k+1}\frac{\bar{N}_c}{N_f}\right)\\
Y:&\quad \left(1, 1, 0, \frac{2}{k+1}\right),\\
M_j:&\quad \left(N_f, \bar{N}_f, 0, 2 - \frac{4}{k+1}\frac{N_c}{N_f} + \frac{2}{k+1}(j-1)\right)
\end{aligned}$$

where we denoted $\bar{N}_c = kN_f - N_c$.

The chiral rings and their duality map were studied in [18]. Following [18] we define the baryon-like operators as follows

$$\begin{aligned}
&B^{i_1^{(1)}\ldots i_{n_1}^{(1)},\ldots,i_1^{(k)}\ldots i_{n_k}^{(k)}} = \\
&\quad = \epsilon^{\alpha_1^{(1)}\ldots\alpha_{n_1}^{(1)}\ldots\alpha_1^{(k)}\ldots\alpha_{n_k}^{(k)}} \left(Q^{i_1^{(1)}}_{\alpha_1^{(1)}}\cdots Q^{i_{n_1}^{(1)}}_{\alpha_{n_1}^{(1)}}\right)\cdots\left((X^{k-1}Q)^{i_1^{(k)}}_{\alpha_1^{(k)}}\cdots(X^{k-1}Q)^{i_{n_k}^{(k)}}_{\alpha_{n_k}^{(k)}}\right),
\end{aligned} \tag{4.4}$$

where $\sum_i n_i = N_c$. The duality map relating operators B with the analogously defined dual baryon operators $\bar{B}$ is

$$B^{i_1^{(1)}\dots i_{n_1}^{(1)},\dots,i_1^{(k)}\dots i_{n_k}^{(k)}} = P\left(\prod_{i=1}^{k}\frac{1}{\bar{n}_i!}\right)(-s_0)^{\frac{kN_f}{2}}(\mu^2)^{\frac{-\bar{N}_c}{2}}\Lambda^{\frac{k}{2}(2N_c-N_f)}\times$$
$$\times\epsilon^{i_1^{(1)}\dots i_{n_1}^{(1)},t_1^{(k)}\dots t_{\bar{n}_k}^{(k)}}\cdots\epsilon^{i_1^{(k)}\dots i_{i_k}^{(k)},t_1^{(1)}\dots t_{\bar{n}_1}^{(1)}}\bar{B}_{t_1^{(k)}\dots t_{\bar{n}_1}^{(k)}\dots t_1^{(1)}\dots t_{\bar{n}_k}^{(1)}} \tag{4.5}$$

where $\bar{n}_l = N_f - n_{k+1-l}$, $l = 1,\dots,k$, $P^2 = (-1)^{\frac{k(k-1)}{2}N_f - N_c}$, μ is an auxiliary scale parameter. In addition to baryons the chiral ring contains mesons $(M_j)^i_l = \tilde{Q}_l X^{j-1} Q^i$ and the generators $\mathrm{Tr}\, X^j$ ($\mathrm{Tr}\, Y^j$), $j = 2,\dots,k$. Notice that from equations of motion we also have

$$X^k - \frac{1}{N_c}\mathrm{Tr}\, X^k = \mathrm{Q} - \mathrm{exact}\,, \qquad Y^k - \frac{1}{N_c}\mathrm{Tr}\, Y^k = \mathrm{Q} - \mathrm{exact}\,. \tag{4.6}$$

Similarly with the discussion of $\mathrm{SU}(N_c)$ SQCD one can easily identify the holomorphic currents corresponding to the baryon symmetry $\mathrm{U}(1)_B$ and the spin-2 currents T_{zz}. The associated central charges match in the electric and magnetic theories since they are proportional to appropriate 't Hooft anomalies.

A new element that appears in the Kutasov-Schwimmer model is an additional structure that imposes non-trivial conditions on the duality map of the chiral ring. In the case when Σ_2 is a torus T^2 with a flat metric one can also define *generalized* baryon currents [4]

$$j^j_z[Q,\tilde{Q},X] = 2\pi\int_{T^2}(-D_z\bar{Q}_i X^j Q^i + \bar{\psi}_i X^j\psi_{\bar{w}} + \bar{\psi}_i j X^{j-1}\chi_{\bar{w}}Q^i) - (Q^i \to \tilde{Q}_i,\ \psi^i \to \tilde{\psi}_i)\,. \tag{4.7}$$

Here $\chi_{\bar{w}}$ stands for the fermionic superpartner of X. Similarly we define generalized baryon current $j^j_z[q,\tilde{q},Y]$ on the magnetic side. Since the Euler characteristic vanishes for torus the central terms in the OPE of these currents also vanish.

As discussed in the previous section, these currents act as vector fields on baryon operators B and $\bar{B}$. In what follows we study consistency of this action with eq. (4.5) and the duality map determined in [18].

Let us denote B defined in eq. (4.4) as $B[n_1,\dots,n_k]$. Consider the action of $j^j_z[Q,\tilde{Q},X]$ on an arbitrary factor in eq. (4.4)

$$(X^l Q)^s_\lambda \to (X^{l+j}Q)^s_\lambda\,. \tag{4.8}$$

If $l + j < k$ this action corresponds to

$$B[n_1,\dots,n_l,\dots,n_{l+j},\dots,n_k] \to B[n_1,\dots,n_l - 1,\dots,n_{l+j}+1,\dots,n_k]\,. \tag{4.9}$$

Thus, the baryon $B[n_1,\dots,n_k]$ transforms into a sum of baryons with a different set of n_l under the action of $j^j_z[Q,\tilde{Q},X]$. For consistency with eq. (4.5), according to eq. (4.4) one should have in the magnetic theory

$$\bar{B}[\bar{n}_1,\dots,\bar{n}_{k+1-l-j},\dots,\bar{n}_{k+1-l},\dots,\bar{n}_k] \to \bar{B}[\bar{n}_1,\dots,\bar{n}_{k+1-l-j}-1,\dots,\bar{n}_{k+1-l}+1,\dots,\bar{n}_k]\,.$$

[4]The integrand in this equation is not a $(1,1)$-form on an arbitrary Σ_2. On a torus this is not important since it can be adjusted by using a constant 1-form.

Indeed, one can see that $j_z^j[q,\tilde{q},Y]$ acts on a factor $(X^{k+1-l-j}Q)^s_\lambda$ as

$$(Y^{k+1-l-j}q)^s_\lambda \to (Y^{k+1-l}q)^s_\lambda \,. \tag{4.10}$$

We will show that this action produces $\bar{B}$ operators that exactly match eq. (4.9).

First, one has to check that the numbers of baryon operators and phase factors match for B and $\bar{B}$ on the right hand side. Let us denote by $[n_l]$ ($[\bar{n}_l]$) the set of flavor indices that are assigned to factors X^lQ (Y^lq) in the electric (magnetic) theory. Notice that the transformation (4.8) for a factor with a flavor index in $[n_l]\cap[n_{l+j}]$ does not produce any baryon operator because of antisymmetrization of the color indices in eq. (4.4). We denote the set of available flavor indices in $[n_l]$ (i.e. indices corresponding to the factors that are not annihilated by (4.8)) as $[\nu_l] = [n_l] - ([n_l]\cap[n_{l+j}])$, and ν_l is the number of elements in $[\nu_l]$. Obviously, $\nu_l \le n_l$. On the magnetic side the set of flavor indices in $[\bar{n}_{k+1-l-j}]$ may also overlap with that of $[\bar{n}_{k+1-l}]$. The action of $j_z^j[q,\tilde{q},Y]$ on the factors with flavor indices in $[\bar{n}_{k+1-l-j}]\cap[\bar{n}_{k+1-l}]$ annihilates the operator $\bar{B}$. The actual number of produced baryon operators equals ν_l. An easy way to see this is to consider the reduced sets $[n_l'] = [n_l] - [n_{l+j}]$, $[n_{l+j}'] = [n_{l+j}] - [n_l]$, $[\bar{n'}_{k+1-l}] = [\bar{n}_{k+1-l}] - [\bar{n}_{k+1-l-j}]$ and $[\bar{n'}_{k+1-l-j}] = [\bar{n}_{k+1-l-j}] - [\bar{n}_{k+1-l}]$. Since $[n_l']\cap[n_{l+j}'] = [\bar{n'}_{k+1-l}]\cap[\bar{n'}_{k+1-l-j}] = \emptyset$ and $[n_l']\cup[n_{l+j}'] = [\bar{n'}_{k+1-l}]\cup[\bar{n'}_{k+1-l-j}]$ we have $[n_l'] = [\bar{n'}_{k+1-l-j}]$. Hence, the set of available flavor indices on the electric side coincides with those on the magnetic side. Thus, under the action of j_z we get equal numbers of baryons in the electric and magnetic theories.

Let us analyze how the flavor indices map for each of the baryons. Originally on the magnetic side we had ϵ factors carrying flavor indices

$$\cdots \epsilon^{i_1^{(l+j)}\ldots i_{n_{l+j}}^{(l+j)},t_1^{(k+1-l-j)}\ldots p\ldots t_{\bar{n}_{k+1-l-j}}^{(k+1-l-j)}} \cdots \epsilon^{i_1^{(l)}\ldots p\ldots i_{n_l}^{(l)},t_1^{(k+1-l)}\ldots t_{\bar{n}_{k+1-l}}^{(k+1-l)}} \cdots \,. \tag{4.11}$$

Let us consider the action of $j_z[q,\tilde{q},Y]$ on the factor $Y^{k+1-l-j}q$ with a flavor index p. Under this action the index p moves from the set of magnetic flavor indices $[\bar{n}_{k+1-l-j}]$ into the set of electric flavor indices $[n_{l+j}]$, and simultaneously it moves from $[n_l]$ into $[\bar{n}_{k+1-l}]$. This exactly corresponds to what happens with the factor X^lQ with the flavor index p. Since a baryon operator is uniquely defined by specifying the corresponding sets of flavor indices, we see that the flavor index structure in the relation (4.5) is supported by this action.

Now let us compute the sign factors that are due to a rearrangement of color and flavor indices after the action of $j_z[q,\tilde{q},Y]$. In order to simplify the discussion, let us choose the convention that the unsummed flavor indices in eq. (4.4) are in the ascending order. It is easy to see that due to the rearrangement of color indices on the electric side the produced baryon acquires a factor $(-1)^{P_{el}}$, where

$$P_{el} = n_l - i^{(l)}(p) + i^{(l+j)}(p) - 1 + \sum_{r=l+1}^{l+j-1} n_r \,.$$

Here $i^{(l)}(p)$ stands for the position of the index p in the ordered set of indices $i^{(l)}$. For the corresponding baryon on the magnetic side one has a factor $(-1)^{P_{\rm mag}}$ where

$$P_{\rm mag} = \left(\bar{n}_{k+1-l-j} - t^{(k+1-l-j)}(p) + t^{(k+1-l)}(p) - 1 + \sum_{r=k+1-l-j+1}^{k+1-l-1} \bar{n}_r \right) + \\ + \left(n_{l+j} - i^{(l+j)}(p) + t^{(k+1-l-j)}(p) - 1 + n_l - i^{(l)}(p) + t^{(k+1-l)}(p) - 1 \right).$$

The terms in the first bracket come from the rearrangement of the color indices while the second bracket is due to the rearrangement of flavor indices in the ϵ factors (4.11). Thus, the relative factor

$$(-1)^{P_{el} - P_{\rm mag}} = (-1)^{jN_f} \tag{4.12}$$

depends only on j but not on the details of the index rearrangements. This factor can be reproduced if we include

$$(-1)^{N_f \sum_l l n_l} \tag{4.13}$$

into the factor P in the baryon operator mapping (4.4). We will show that this assumption agrees with the mass flow analysis in [18] (section (5.1)).

As pointed out in [18] there is a sign ambiguity the duality map in eq. (4.5) due to non-anomalous global symmetries (therefore the authors of [18] focused on phase factor for $B\tilde{B}$ that is invariant). However, relative phases of baryons with the same flavor indices does not change under phase rotations of quarks superfields. The factor (4.13) controls such relative phases. By introducing a mass term $m\tilde{Q}Q$ in the electric theory the number of flavors reduces $N_f \to N_f - 1$, and only baryons without a flavor index N_f survive in the infrared limit. This flow corresponds [18] to Higgsing $\mathrm{SU}(\bar{N}_c) \to \mathrm{SU}(\bar{N}_c - k)$ in the magnetic theory with non-zero vevs

$$\tilde{q}_\alpha^{\;N_f} = \delta_{\alpha,1} \left(\frac{m\mu^2}{-s_0} \right)^{\frac{1}{k+1}}, \qquad q^\alpha_{N_f} = \delta^{\alpha,k} \left(\frac{m\mu^2}{-s_0} \right)^{\frac{1}{k+1}}, \qquad Y^\alpha_\beta = \delta^\alpha_{\beta-1} \left(\frac{m\mu^2}{-s_0} \right)^{\frac{1}{k+1}}, \tag{4.14}$$

and hence, the number of flavors in the magnetic theory also reduces $N_f \to N_f - 1$. In the expression for a magnetic baryon $\bar{B}$ the broken color indices $\alpha = 1, \ldots, k$ are taken by the factors $Y^j q^{N_f}$. The rearrangements of color and flavor indices that is needed to bring baryons to the standard form, give the factors

$$(-1)^{\frac{k(k-1)}{2} + (N_f - 1)\frac{k(k+1)}{2} + \sum_l l n_l} \qquad \text{and} \qquad (-1)^{(N_f-1)\frac{k(k+1)}{2} + \sum_l l n_l}$$

for $\tilde{\bar{B}}[\bar{n}_1, \ldots, \bar{n}_k]$ and $\bar{B}[\bar{n}_1, \ldots, \bar{n}_k]$ respectively. Thus, the change of the phase in the map for $B\tilde{B}$ is $(-1)^{\frac{k(k-1)}{2}}$ as stated in [18]. The factor $(-1)^{\sum_l l n_l}$ is exactly equal to the change of (4.13) under $N_f \to N_f - 1$.

If $l + j \geq k$ eq. (4.8) is equivalent to

$$(X^l Q)^s_\lambda \to (X^{l+j-k} Q)^s_\lambda \, \frac{1}{N_c} \operatorname{Tr} X^k \tag{4.15}$$

and hence,

$$B[n_1,\ldots,n_{l+j-k},\ldots,n_l,\ldots,n_k] \to B[n_1,\ldots,n_{l+j-k}+1,\ldots,n_l-1,\ldots,n_k]\frac{1}{N_c}\,\mathrm{Tr}\,X^k\,.$$

Here the numbers n_l jump under the action of $j_z^j[Q,\tilde{Q},X]$. Similarly with the previous case, the numbers of produced baryons and phase factors match in the electric and magnetic theories. The only new element is the presence of different factors $\frac{1}{N_c}\,\mathrm{Tr}\,X^k$ and $\frac{1}{\bar{N}_c}\,\mathrm{Tr}\,Y^k$ in the formulas for B and $\bar{B}$ transformations respectively. However, according to the results of [18] in the chiral ring one has

$$\frac{1}{N_c}\,\mathrm{Tr}\,X^k = \frac{1}{\bar{N}_c}\,\mathrm{Tr}\,Y^k\,. \tag{4.16}$$

Thus, the currents $j_z^j[Q,\tilde{Q},X]$ and $j_z^j[q,\tilde{q},Y]$ act in the same way on the baryon operators up to a sign factor. These currents annihilate the meson operators and $\mathrm{Tr}X^l$, $\mathrm{Tr}Y^l$. Thus, we conclude that

$$j_z^j[Q,\tilde{Q},X] = j_z^j[q,\tilde{q},Y] + Q-\text{exact terms}\,. \tag{4.17}$$

Notice that the identity (4.16) obtained in [18] from very different arguments was crucial for the consistency of this identification. One can view this fact as an additional argument supporting the duality picture developed in [18].

One can extend this analysis to the currents

$$j_z^{j,a}[Q,\tilde{Q},X] = 2\pi\int_{T^2}(-D_z\bar{Q}T^aX^jQ + \bar{\psi}T^aX^j\psi_{\bar{w}} + \bar{\psi}T^a jX^{j-1}\chi_{\bar{w}}Q) \tag{4.18}$$

those are not singlets with respect to the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ flavor group. Here T^a stand for a generator of $\mathrm{SU}(N_f)$, and flavor and color indices are implicit. This current corresponds to the following transformations

$$\delta_j^a Q \to T^aX^jQ\,. \tag{4.19}$$

Let us determine the corresponding transformation on the magnetic side. The action on the magnetic quarks reads as

$$\delta_j^a q \to T^aY^jq \tag{4.20}$$

while for mesons one has

$$\begin{aligned}\delta_j^a M_l &= -T^aM_{l+j}\,, \qquad l+j\le k\,,\\ \delta_j^a(M_l)_f^{f'} &= -(M_{l+j-k})_p^{f'}(N_{2k-l}N_{k-l}^{-1})_f^p\,, \qquad l+j>k\,,\end{aligned}$$

where in the second line we show explicitly the $\mathrm{SU}(N_f)\times\mathrm{SU}(N_f)$ flavor f,f',p indices and

$$(N_l)_f^{f'} \equiv \tilde{q}^{f'}Y^lq_f\,.$$

Notice that in the chiral ring the transformation at $l+j>k$ reduces to

$$\delta_j^a M_l \to -\frac{1}{\bar{N}_c}\,\mathrm{Tr}\,Y^kT^aM_{l+j-k}\,, \qquad l+j\ge k+1\,.$$

Thus, the total current reads as

$$j^{j,a}_{tot,z}[q,\bar{q},M,\bar{M},Y] = j^{j,a}_z[q,\tilde{q},Y] + m^{j,a}_z[M,\bar{M},Y]\,, \tag{4.21}$$

where

$$m^{j,a}_z[M,\bar{M},Y] = -2\pi\int_{T^2}\left(-\sum_l \partial_z\bar{M}_l\delta^a_j M_l + \text{fermionic terms}\right). \tag{4.22}$$

The fermionic terms in eq. (4.22) can be restored by using eq. (3.1). Clearly, the action of $j^{j,a}_{tot,z}[q,\bar{q},M,\bar{M},Y]$ on meson operators supports the duality map $M_l = \tilde{Q}X^lQ$.

The problem with eq. (4.21) is that the terms corresponding to $l+j>k$ are singular in chiral fields. This fact apparently implies that one has to extend the ultraviolet off-shell formulation of magnetic theory by introducing additional chiral superfields. These additional degrees of freedom should become massive in the infrared limit. We will assume that the current (4.21) is well defined in the context of operator product expansion.

Consider the action of $j^{j,a}_z[Q,\tilde{Q},X]$ on the baryon $B[n_1,\ldots,n_k]$. For a diagonal generator T^a the analysis is similar to the case of the singlet current. Therefore, it is sufficient to consider only non-diagonal generators $(S^{(p,q)})_{ij} = \delta_{ip}\delta_{jq}+\delta_{jp}\delta_{iq}$ and $(A^{(p,q)})_{ij} = i\delta_{ip}\delta_{jq} - i\delta_{jp}\delta_{iq}$, $p\neq q$. Consider a factor X^lQ^p in $B[n_1,\ldots,n_k]$. The action of a non-diagonal current on it may be decomposed into the action of a diagonal current and a change of the index $p\to q$. Assuming that this action does not annihilate the baryon operator one has $p\in[n_l]$ and $q\notin[n_{l+j}]$. Hence, $p\notin[\bar{n}_{k+1-l}]$ and $q\in[\bar{n}_{k+1-l-j}]$. Thus, moving the index p from $[n_l]$ to $[n_{l+j}]$ and replacing it with q corresponds to moving the index q from $[\bar{n}_{k+1-l-j}]$ to $[\bar{n}_{k+1-l}]$ and replacing it with p. This is obviously a one-to-one correspondence that supports the relation (4.5). Therefore, we have the same sets of baryons produced in the electric and magnetic theories.

The analysis of the phase factors is similar to the case of diagonal currents. The only subtlety here is that the currents corresponding to the generators $A^{(p,q)}$ produce different additional factors i and $-i$ on the electric and magnetic side respectively. This difference in sign is accounted for by the fact that the electric and magnetic quarks belong to the N_f and $\bar{N}_f$ representations of the group $\mathrm{SU}(N_f)$ respectively. Thus, we have

$$j^{j,a}_z[Q,\tilde{Q},X] = j^{j,a}_{tot,z}[q,\bar{q},M,\bar{M},Y] + Q-\text{exact terms}\,. \tag{4.23}$$

Hence, the algebras generated by the holomorphic currents match in the electric and magnetic theories.

5. Discussion

In this paper we studied the algebra generated by non-local holomorphic currents in twisted supersymmetric theories. The central charge in the OPE of the holomorphic spin-2 current with itself coincides with the infrared value of the a central charge, i.e. the coefficient of the Euler density in the trace anomaly. Holomorphic spin-1 currents related to the global symmetries have central charges proportional to 't Hooft anomalies. We also discussed the algebra of generalized holomorphic currents corresponding to non-linear transformations of chiral fields.

The holomorphic algebras have interesting applications in the context of electric-magnetic duality. In particular, central charges of holomorphic currents in the electric theory should coincide with the corresponding central charges in the magnetic theory. Matching generalized holomorphic currents in the electric and magnetic theories imposes non-trivial conditions on the duality map of the chiral ring. We considered in some detail the $\mathrm{SU}(N_c)$ supersymmetric QCD and the adjoint supersymmetric QCD (Kutasov-Schwimmer models), and found agreement of the holomorphic structure with the electric-magnetic duality. However, we found that in the magnetic description the generalized non-abelian flavor currents are not polynomial.

This analysis can be generalized to other models that exhibit electric-magnetic duality. An interesting example is the adjoint supersymmetric QCD discussed above with vanishing superpotential. As discussed in [18], when $N_f \lesssim 2N_c$ the theory flows to a non-abelian Coulomb phase in the infrared. There is a one parameter set of R-symmetries. The physical R-symmetry can be determined by using the results of [26]. The renormalization group flows and phase structure of this theory has been studied in details in [25]. However, there is no known magnetic description.

This theory can be twisted by using the R-current that corresponds to the following R-charges of the chiral fields: $R(X) = 0$, $R(Q) = 0$ and $R(\tilde{Q}) = 1$. Since there is no superpotential the chiral ring is generated by $\mathrm{Tr}X^j$, $j = 0, \ldots$ and baryon operators defined similarly with eq. (4.4). However, in this case there is no restrictions on powers of X. For this choice of the twisting R-current the field X has zero spin, and the generalized holomorphic currents $j^j_{\bar{z}}[Q, \tilde{Q}, X]$ can be defined for arbitrary surface Σ_2. These holomorphic currents generate an *infinite* algebra with non-trivial "central" operators $\mathrm{Tr}X^j$. If a magnetic description of the theory exists, the magnetic theory has to possess a holomorphic algebra isomorphic to that on the electric side. Then this isomorphism may give information on a duality map for the chiral ring.

Notice that in models where *accidental* symmetries appear in the infrared limit the actual holomorphic algebra is larger than that visible in the ultraviolet regime. This happens when some chiral operators reach the unitarity bound under the renormalization group flow and decouple. For example, an accidental symmetry corresponding to decoupling of $Q\tilde{Q}$ appears in the presence of a superpotential (4.2) for $k = 2$ and $N_f < N_c$ in the Kutasov-Schwimmer model. Models with accidental symmetries may be an interesting playground for applications of holomorphic currents.

It would be also interesting to generalize the construction of holomorphic algebras to deformed models where R-symmetry is broken by a superpotential. By promoting the coupling in the superpotential to chiral superfields one can still twist the theory. Such a generalization could allow considering the behavior of holomorphic currents under interpolation between various conformal theories. Also, by integrating over massive fields one can get an effective sigma-model for the coupling constants where the holomorphic algebra should also have a realization. By matching the holomorphic algebras in these two limits could obtain some conditions on the quantum metric on the space of deformations of the model.

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